Solution of Linear Systems

A linear combination of the variables $x_1, x_2, \ldots, x_N$ is a sum

(1) $$a_1x_1 + a_2x_2 + \cdots + a_Nx_N$$

where a_k is the coefficient of x_k for $k = 1, 2, \ldots, N$.

A linear equation in $x_1, x_2, \ldots, x_N$ is obtained by requiring the linear combination in (1) to take on a prescribed value b; that is,

(2) $$a_1x_1 + a_2x_2 + \cdots + a_Nx_N = b.$$

Systems of linear equations arise frequently, and if M equations in N unknowns are given, we write

(3)
$$\begin{array}{lllll}
a_{11}x_1 & + a_{12}x_2 & + \cdots + a_{1N}x_N & = b_1 \\
a_{21}x_1 & + a_{22}x_2 & + \cdots + a_{2N}x_N & = b_2 \\
\vdots & \vdots & \vdots & \vdots \\
a_{k1}x_1 & + a_{k2}x_2 & + \cdots + a_{kN}x_N & = b_k \\
\vdots & \vdots & \vdots & \vdots \\
a_{M1}x_1 & + a_{M2}x_2 & + \cdots + a_{MN}x_N & = b_M.
\end{array}$$

in matrix form

$$AX = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1N} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2N} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{kj} & \cdots & a_{kN} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{M1} & a_{M2} & \cdots & a_{Mj} & \cdots & a_{MN} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_j \\ \vdots \\ x_N \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_j \\ \vdots \\ b_M \end{bmatrix} = B$$

- AX=B can be transformed into an equivalent system which may be easier to solve.
- Equivalent system has the same solution as the original system.
- Allowable operations during the transformation are:

(1) Interchanges: The order of two equations can be changed.

(2) Scaling: Multiplying an equation by a nonzero constant.

(3) Replacement: An equation can be replaced by the sum of itself and a nonzero multiple of any other equation.

Example 3.15. Find the parabola $y = A + Bx + Cx^2$ that passes through the three points $(1, 1)$, $(2, -1)$, and $(3, 1)$.

For each point we obtain an equation relating the value of x to the value of y. The result is the linear system

$$
\begin{aligned}
A + B + C &= 1 && \text{at } (1,1) \\
A + 2B + 4C &= -1 && \text{at } (2,-1) \\
A + 3B + 9C &= 1 && \text{at } (3,1).
\end{aligned} \tag{4}
$$

The variable A is eliminated from the second and third equations by subtracting the first equation from them. This is an application of the replacement transformation (3), and the resulting equivalent linear system is

$$
\begin{aligned}
A + B + C &= 1 \\
B + 3C &= -2 \\
2B + 8C &= 0.
\end{aligned} \tag{5}
$$

The variable B is eliminated from the third equation in (5) by subtracting from it two times the second equation. We arrive at the equivalent upper-triangular system:

$$
\begin{aligned}
A + B + C &= 1 \\
B + 3C &= -2 \\
2C &= 4.
\end{aligned} \tag{6}
$$

The back-substitution algorithm is now used to find the coefficients $C = 4/2 = 2$, $B = -2 - 3(2) = -8$, and $A = 1 - (-8) - 2 = 7$, and the equation of the parabola is $y = 7 - 8x + 2x^2$. ■

Gaussian Elimination for solving $[A][X] = [C]$

consists of 2 steps

1. Forward Elimination of unknowns

The goal of Forward Elimination is to transform the coefficient matrix into an Upper Triangular Matrix

$$
\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix}
$$

2. Back Substitution

The goal of Back Substitution is to solve each of the equations using the upper triangular matrix.

Gaussian Elimination

The augmented matrix is [A|B] and the system AX = B is represented as follows:

$$[A|B] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1N} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2N} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{N1} & a_{N2} & \cdots & a_{NN} & b_N \end{array}\right]. \tag{7}$$

The system $\boldsymbol{AX} = \boldsymbol{B}$, with augmented matrix given in (7), can be solved by performing row operations on the augmented matrix $[\boldsymbol{A}|\boldsymbol{B}]$. The variables x_k are placeholders for the coefficients and can be omitted until the end of the calculation.

Theorem 3.8 (Elementary Row Operations). The following operations applied to the augmented matrix (7) yield an equivalent linear system.

(8) Interchanges: The order of two rows can be changed.

(9) Scaling: Multiplying a row by a nonzero constant.

(10) Replacement: The row can be replaced by the sum of that row and a nonzero multiple of any other row; that is: $\text{row}_r = \text{row}_r - m_{rp} \times \text{row}_p$.

Example 3.16

$$\begin{aligned} x_1 + 2x_2 + x_3 + 4x_4 &= 13 \\ 2x_1 + 0x_2 + 4x_3 + 3x_4 &= 28 \\ 4x_1 + 2x_2 + 2x_3 + x_4 &= 20 \\ -3x_1 + x_2 + 3x_3 + 2x_4 &= 6. \end{aligned}$$

$$\begin{array}{l} \text{pivot} \rightarrow \\ m_{21} = 2 \\ m_{31} = 4 \\ m_{41} = -3 \end{array} \left[\begin{array}{cccc|c} \underline{1} & 2 & 1 & 4 & 13 \\ 2 & 0 & 4 & 3 & 28 \\ 4 & 2 & 2 & 1 & 20 \\ -3 & 1 & 3 & 2 & 6 \end{array}\right]$$

$$\begin{array}{l} \\ \text{pivot} \rightarrow \\ m_{32} = 1.5 \\ m_{42} = -1.75 \end{array} \left[\begin{array}{cccc|c} 1 & 2 & 1 & 4 & 13 \\ 0 & \underline{-4} & 2 & -5 & 2 \\ 0 & -6 & -2 & -15 & -32 \\ 0 & 7 & 6 & 14 & 45 \end{array}\right]$$

$$\begin{array}{l} \\ \\ \text{pivot} \rightarrow \\ m_{43} = -1.9 \end{array} \left[\begin{array}{cccc|c} 1 & 2 & 1 & 4 & 13 \\ 0 & -4 & 2 & -5 & 2 \\ 0 & 0 & \underline{-5} & -7.5 & -35 \\ 0 & 0 & 9.5 & 5.25 & 48.5 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 1 & 2 & 1 & 4 & 13 \\ 0 & -4 & 2 & -5 & 2 \\ 0 & 0 & -5 & -7.5 & -35 \\ 0 & 0 & 0 & -9 & -18 \end{array}\right] \qquad x_4 = 2, \quad x_3 = 4, \quad x_2 = -1, \quad x_1 = 3.$$

Forward Elimination

Linear Equations

A set of *n* equations and *n* unknowns

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots + a_{2n}x_n = b_2$$

$$\vdots \qquad \vdots$$

$$a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \ldots + a_{nn}x_n = b_n$$

Forward Elimination

Transform to an Upper Triangular Matrix

Step 1: Eliminate x_1 in 2nd equation using equation 1 as the pivot equation (pivot row)

$$\left[\frac{Eqn1}{a_{11}}\right] \times (a_{21})$$

Which will yield

$$a_{21}x_1 + \frac{a_{21}}{a_{11}}a_{12}x_2 + \ldots + \frac{a_{21}}{a_{11}}a_{1n}x_n = \frac{a_{21}}{a_{11}}b_1$$

a11:pivot element, row 1:pivot row

Forward Elimination

Zeroing out the coefficient of x_1 in the 2nd equation.

Subtract this equation from 2nd equation

$$\left(a_{22} - \frac{a_{21}}{a_{11}} a_{12}\right) x_2 + \ldots + \left(a_{2n} - \frac{a_{21}}{a_{11}} a_{1n}\right) x_n = b_2 - \frac{a_{21}}{a_{11}} b_1$$

Or

$$a'_{22} x_2 + \ldots + a'_{2n} x_n = b'_2$$

Where

$$a'_{22} = a_{22} - \frac{a_{21}}{a_{11}} a_{12}$$

$$\vdots$$

$$a'_{2n} = a_{2n} - \frac{a_{21}}{a_{11}} a_{1n}$$

Forward Elimination

Repeat this procedure for the remaining equations to reduce the set of equations as

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n &= b_1 \\
a'_{22}x_2 + a'_{23}x_3 + \ldots + a'_{2n}x_n &= b'_2 \\
a'_{32}x_2 + a'_{33}x_3 + \ldots + a'_{3n}x_n &= b'_3 \\
\vdots \qquad \vdots \qquad\quad \vdots & \\
a'_{n2}x_2 + a'_{n3}x_3 + \ldots + a'_{nn}x_n &= b'_n
\end{aligned}$$

Forward Elimination

Step 2: Eliminate x_2 in the 3rd equation.

Equivalent to eliminating x_1 in the 2nd equation using equation 2 as the pivot equation.

$$Eqn3 - \left[\frac{Eqn2}{a_{22}} \right] \times (a_{32})$$

Forward Elimination

This procedure is repeated for the remaining equations to reduce the set of equations as

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n &= b_1 \\
a'_{22}x_2 + a'_{23}x_3 + \ldots + a'_{2n}x_n &= b'_2 \\
a''_{33}x_3 + \ldots + a''_{3n}x_n &= b''_3 \\
\vdots \qquad\qquad & \vdots \\
a''_{n3}x_3 + \ldots + a''_{nn}x_n &= b''_n
\end{aligned}$$

Forward Elimination

Continue this procedure by using the third equation as the pivot equation and so on.

At the end of (n-1) Forward Elimination steps, the system of equations will look like:

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n &= b_1 \\
a'_{22}x_2 + a'_{23}x_3 + \dots + a'_{2n}x_n &= b'_2 \\
a''_{33}x_3 + \dots + a''_{n}x_n &= b''_3 \\
&\vdots \\
a_{nn}^{(n-1)}x_n &= b_n^{(n-1)}
\end{aligned}
$$

Forward Elimination

At the end of the Forward Elimination steps

$$
\begin{bmatrix}
a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\
 & a'_{22} & a'_{23} & \cdots & a'_{2n} \\
 & & a''_{33} & \cdots & a''_{3n} \\
 & & & \vdots & \vdots \\
 & & & & a_{nn}^{(n-1)}
\end{bmatrix}
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}
=
\begin{bmatrix} b_1 \\ b'_2 \\ b''_3 \\ \vdots \\ b_n^{(n-1)} \end{bmatrix}
$$

Back Substitution

The goal of Back Substitution is to solve each of the equations using the upper triangular matrix.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Example of a system of 3 equations

Back Substitution

Start with the last equation because it has only one unknown

$$x_n = \frac{b_n^{(n-1)}}{a_{nn}^{(n-1)}}$$

Solve the second from last equation $(n-1)^{th}$ using x_n solved for previously.

This solves for x_{n-1}.

Back Substitution

Representing Back Substitution for all equations by formula

$$x_i = \frac{b_i^{(i-1)} - \sum_{j=i+1}^{n} a_{ij}^{(i-1)} x_j}{a_{ii}^{(i-1)}} \qquad \text{For } i=n-1, n-2, \ldots, 1$$

and

$$x_n = \frac{b_n^{(n-1)}}{a_{nn}^{(n-1)}}$$

Potential Pitfalls

-Division by zero: May occur in the forward elimination steps.

-Round-off error: Prone to round-off errors.

Improvements

Increase the number of significant digits

- Decreases round off error
- Does not avoid division by zero

Gaussian Elimination with Pivoting

- Avoids division by zero
- Reduces round off error

Division by zero

Consider the system of equations

$$\begin{aligned} 2x + 6y + 10z &= 0, \\ x + 3y + 3z &= 2, \\ 3x + 14y + 28z &= -8. \end{aligned}$$

The augmented matrix is

$$\left[\begin{array}{ccc|c} 2 & 6 & 10 & 0 \\ 1 & 3 & 3 & 2 \\ 3 & 14 & 28 & -8 \end{array}\right].$$

The first stage of elimination gives

$$\left[\begin{array}{ccc|c} 2 & 6 & 10 & 0 \\ 0 & 0 & -2 & 2 \\ 0 & 5 & 13 & -8 \end{array}\right].$$

We are unable to continue, unless we interchange the second and third rows:

$$\left[\begin{array}{ccc|c} 2 & 6 & 10 & 0 \\ 0 & 5 & 13 & -8 \\ 0 & 0 & -2 & 2 \end{array}\right].$$

In this simple example, after the row pivoting, no further elimination is required.
By back substitution,

$$x_3 = -1;$$

$$x_2 = \frac{1}{5}[-13(-1) - 8] = 1;$$

$$x_1 = \frac{1}{2}[-6(1) - 10(-1) + 0] = 2.$$

Trivial pivoting

Pivoting to Avoid $a_{pp}^{(p)} = 0$

If $a_{pp}^{(p)} = 0$, row p cannot be used to eliminate the elements in column p below the main diagonal. It is necessary to find row k, where $a_{kp}^{(p)} \neq 0$ and $k > p$, and then interchange row p and row k so that a nonzero pivot element is obtained. This process is called *pivoting*, and the criterion for deciding which row to choose is called a *pivoting strategy*. The ***trivial pivoting*** strategy is as follows. If $a_{pp}^{(p)} \neq 0$, do not switch rows. If $a_{pp}^{(p)} = 0$, locate the first row below p in which $a_{kp}^{(p)} \neq 0$ and switch rows k and p. This will result in a new element $a_{pp}^{(p)} \neq 0$, which is a nonzero pivot element.

Round-off error

Example 3.17. The values $x_1 = x_2 = 1.000$ are the solutions to

$$\begin{aligned} 1.133x_1 + 5.281x_2 &= 6.414 \\ 24.14x_1 - 1.210x_2 &= 22.93. \end{aligned} \tag{12}$$

$$\begin{aligned} 1.133x_1 + 5.281x_2 &= 6.414 \\ -113.7x_2 &= -113.8. \end{aligned}$$

Back substitution is used to compute $x_2 = -113.8/(-113.7) = 1.001$, and $x_1 = (6.414 - 5.281(1.001))/(1.133) = (6.414 - 5.286)/1.133 = 0.9956$. ■

$$\begin{aligned} 24.14x_1 - 1.210x_2 &= 22.93 \\ 1.133x_1 + 5.281x_2 &= 6.414. \end{aligned}$$

$$\begin{aligned} 24.14x_1 - 1.210x_2 &= 22.93 \\ 5.338x_2 &= 5.338. \end{aligned}$$

Back substitution is used to compute $x_2 = 5.338/5.338 = 1.000$, and $x_1 = (22.93 + 1.210(1.000))/24.14 = 1.000$. ■

Pivoting to reduce error

- Partial pivoting
- Scaled partial pivoting

Partial Pivoting

Gaussian Elimination with partial pivoting applies row switching to normal Gaussian Elimination.

How?

At the beginning of the k^{th} step of forward elimination, find the maximum of

$$|a_{kk}|, |a_{k+1,k}|, \ldots\ldots\ldots\ldots, |a_{nk}|$$

(*find max of all elements in the column on or below the main diagonal*)

If the maximum of the values is $|a_{pk}|$ In the p^{th} row, $k \le p \le n,$

then switch rows p and k.

Partial Pivoting

What does it Mean?

Gaussian Elimination with Partial Pivoting ensures that each step of Forward Elimination is performed with the pivoting element $|a_{kk}|$ having the largest absolute value.

Partial Pivoting: Example

Consider the system of equations

$$10x_1 - 7x_2 = 7$$
$$-3x_1 + 2.099x_2 + 3x_3 = 3.901$$
$$5x_1 - x_2 + 5x_3 = 6$$

In matrix form

$$\begin{bmatrix} 10 & 7 & 0 \\ -3 & 2.099 & 6 \\ 5 & -1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 3.901 \\ 6 \end{bmatrix}$$

Solve using Gaussian Elimination with Partial Pivoting using five significant digits with chopping

Partial Pivoting: Example

Forward Elimination: Step 1

Examining the values of the first column

|10|, |-3|, and |5| or 10, 3, and 5

The largest absolute value is 10, which means, to follow the rules of Partial Pivoting, we switch row1 with row1.

Performing Forward Elimination

$$\begin{bmatrix} 10 & 7 & 0 \\ -3 & 2.099 & 6 \\ 5 & -1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 3.901 \\ 6 \end{bmatrix} \Rightarrow \begin{bmatrix} 10 & -7 & 0 \\ 0 & -0.001 & 6 \\ 0 & 2.5 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 6.001 \\ 2.5 \end{bmatrix}$$

Partial Pivoting: Example

Forward Elimination: Step 2

Examining the values of the first column

|-0.001| and |2.5| or 0.0001 and 2.5

The largest absolute value is 2.5, so row 2 is switched with row 3

Performing the row swap

$$\begin{bmatrix} 10 & -7 & 0 \\ 0 & -0.001 & 6 \\ 0 & 2.5 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 6.001 \\ 2.5 \end{bmatrix} \Rightarrow \begin{bmatrix} 10 & -7 & 0 \\ 0 & 2.5 & 5 \\ 0 & -0.001 & 6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 2.5 \\ 6.001 \end{bmatrix}$$

Partial Pivoting: Example

Forward Elimination: Step 2

Performing the Forward Elimination results in:

$$\begin{bmatrix} 10 & -7 & 0 \\ 0 & 2.5 & 5 \\ 0 & 0 & 6.002 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 2.5 \\ 6.002 \end{bmatrix}$$

Partial Pivoting: Example

Back Substitution

Solving the equations through back substitution

$$\begin{bmatrix} 10 & -7 & 0 \\ 0 & 2.5 & 5 \\ 0 & 0 & 6.002 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 2.5 \\ 6.002 \end{bmatrix}$$

$$x_3 = \frac{6.002}{6.002} = 1$$

$$x_2 = \frac{2.5 - 5x_2}{2.5} = 1$$

$$x_1 = \frac{7 + 7x_2 - 0x_3}{10} = 0$$

Scaled partial pivoting

tations could result in an erroneous answer. The technique of ***scaled partial pivoting*** or equilibrating can be used to further reduce the effect of error propagation. In scaled partial pivoting we search all the elements in column p that lie on or below the main diagonal for the one that is largest relative to the entries in its row. First search rows p through N for the largest element in magnitude in each row, say s_r:

$$s_r = \max\{|a_{rp}|, |a_{rp+1}|, \ldots, |a_{rN}|\} \quad \text{for } r = p,\ p+1,\ \ldots,\ N. \tag{13}$$

The pivotal row k is determined by finding

$$\frac{|a_{kp}|}{s_k} = \max\left\{\frac{|a_{pp}|}{s_p}, \frac{|a_{p+1p}|}{s_{p+1}}, \ldots, \frac{|a_{Np}|}{s_N}\right\}. \tag{14}$$

Now interchange row p and k, unless $p = k$. Again, this pivoting process is designed to keep the relative magnitudes of the elements in the matrix $\boldsymbol{U}$ in Theorem 3.9 the same as those in the original coefficient matrix $\boldsymbol{A}$.

Scaled partial pivoting example

Use Gaussian elimination with scaled partial pivoting to solve

$$\begin{aligned} 2x + 5y + 3z \quad\quad &= 4 \\ 2x + 6y \quad\quad + 4v &= 10 \\ -x - 2y \quad\quad + 1v &= 0 \\ 2y + 9z + 9v &= 13 \end{aligned} \qquad [\mathbf{s} : A : \mathbf{b}] = \left[\begin{array}{c:cccc:c} 5 & 2 & 5 & 3 & 0 & 4 \\ 6 & 2 & 6 & 0 & 4 & 10 \\ 2 & -1 & -2 & 0 & 1 & 0 \\ 9 & 0 & 2 & 9 & 9 & 13 \end{array}\right] \quad \text{(55a)}$$

Solution

We begin the upper-triangulation phase by taking $[\mathbf{s}^{(0)} : A^{(0)} : \mathbf{b}^{(0)}]$ to be the given $[\mathbf{s} : A : \mathbf{b}]$, with $\mathbf{s}$ obtained from A using (54a). Then we do the following.

$m = 1$. The $|a_{i1}|/s_i$ ratios (54b) for the four a_{i1}'s at and below $a_{11} = 2$ in $A^{(0)}$ are

$$\frac{|2|}{5},\ \frac{|2|}{6},\ \frac{|-1|}{2},\ \text{and}\ \frac{|0|}{9}\ \left(\text{largest ratio is } \frac{|-1|}{2}, \text{ for row 3; so } p = 3\right) \quad \text{(55b)}$$

We therefore perform a 1,3-pivot in $[\mathbf{s}^{(0)} : A^{(0)} : \mathbf{b}^{(0)}]$ *before* reducing $\mathrm{col}_1[A^{(0)} : \mathbf{b}^{(0)}]$:

$$\underbrace{\left[\begin{array}{c:cccc:c} 2 & -1 & -2 & 0 & 1 & 0 \\ 6 & 2 & 6 & 0 & 4 & 10 \\ 5 & 2 & 5 & 3 & 0 & 4 \\ 9 & 0 & 2 & 9 & 9 & 13 \end{array}\right]}_{\text{rearranged } [\mathbf{s}^{(0)} : A^{(0)} : \mathbf{b}^{(0)}]} \xrightarrow[\substack{\text{row}_2 - \frac{2}{-1}\text{row}_1 \\ \text{row}_3 - \frac{2}{-1}\text{row}_1 \\ \text{row}_4 - \frac{0}{-1}\text{row}_1}]{} \underbrace{\left[\begin{array}{c:cccc:c} 2 & -1 & -2 & 0 & 1 & 0 \\ 6 & 0 & 2 & 0 & 6 & 10 \\ 5 & 0 & 1 & 3 & 2 & 4 \\ 9 & 0 & 2 & 9 & 9 & 13 \end{array}\right]}_{[\mathbf{s}^{(1)} : A^{(1)} : \mathbf{b}^{(1)}]} \quad \text{(55c)}$$

$m = 2$. The $|a_{i2}|/s_i$ ratios for the three a_{i2}'s at and below $a_{22} = 2$ in $A^{(1)}$ are

$$\frac{|2|}{6},\ \frac{|1|}{5},\ \text{and}\ \frac{|2|}{9}\ \left(\text{largest is } \frac{|2|}{6}, \text{ for row 2; so } p = 2\right) \quad \text{(55d)}$$

Since $p = m$, we reduce $\mathrm{col}_2[A^{(1)} : \mathbf{b}^{(1)}]$ *without* a $2,p$-pivot:

$$[\mathbf{s}^{(1)} : A^{(1)} : \mathbf{b}^{(1)}] \xrightarrow[\substack{\text{row}_3 - \frac{1}{2}\text{row}_2 \\ \text{row}_4 - \frac{2}{2}\text{row}_2}]{} \left[\begin{array}{c:cccc:c} 2 & -1 & -2 & 0 & 1 & 0 \\ 6 & 0 & 2 & 0 & 6 & 10 \\ 5 & 0 & 0 & 3 & -1 & -1 \\ 9 & 0 & 0 & 9 & 3 & 3 \end{array}\right] = [\mathbf{s}^{(2)} : A^{(2)} : \mathbf{b}^{(2)}] \quad \text{(55e)}$$

$m = 3$. The $|a_{i3}|/s_i$ ratios for the two a_{i3}'s at and below $a_{33} = 3$ in $A^{(2)}$ are

$$\frac{|3|}{5}\ \text{and}\ \frac{|9|}{9}\ \left(\text{largest is } \frac{|9|}{9}, \text{ for row 4; } p = 4\right) \quad \text{(55f)}$$

We therefore perform a 3,4-pivot in $[\mathbf{s}^{(2)} : A^{(2)} : \mathbf{b}^{(2)}]$ before reducing the second column. The 3,4-subtract in the rearranged $[\mathbf{s}^{(2)} : A^{(2)} : \mathbf{b}^{(2)}]$ yields $[\mathbf{s}^{(3)} : A^{(3)} : \mathbf{b}^{(3)}] = [\hat{\mathbf{s}} : \hat{U} : \hat{\mathbf{c}}]$:

$$\underbrace{\left[\begin{array}{c:cccc:c} 2 & -1 & -2 & 0 & 1 & 0 \\ 6 & 0 & 2 & 0 & 6 & 10 \\ 9 & 0 & 0 & 9 & 3 & 3 \\ 5 & 0 & 0 & 3 & -1 & -1 \end{array}\right]}_{\text{rearranged } [\mathbf{s}^{(2)} : A^{(2)} : \mathbf{b}^{(2)}]} \xrightarrow[\text{row}_4 - \frac{3}{9}\text{row}_3]{} \underbrace{\left[\begin{array}{c:cccc:c} 2 & -1 & -2 & 0 & 1 & 0 \\ 6 & 0 & 2 & 0 & 6 & 10 \\ 9 & 0 & 0 & 9 & 3 & 3 \\ 5 & 0 & 0 & 0 & -2 & -2 \end{array}\right]}_{[\mathbf{s}^{(3)} : A^{(3)} : \mathbf{b}^{(3)}]} = [\hat{\mathbf{s}} : \hat{U} : \hat{\mathbf{c}}] \quad \text{(55g)}$$

Solving $\hat{U}\mathbf{x} = \hat{\mathbf{c}}$ by backward substitution yields first $\bar{v} = (-2)/(-2) = 1$, then

$$\bar{z} = \frac{3 - 3\bar{v}}{9} = 0; \quad \bar{y} = \frac{10 - (0\bar{z} + 6\bar{v})}{2} = 2; \quad \bar{x} = \frac{0 - (-2\bar{y} + 0\bar{z} + 1\bar{v})}{-1} = -3 \quad \text{(55h)}$$

Since $\hat{U}\mathbf{x} = \hat{\mathbf{c}}$ is equivalent to $A\mathbf{x} = \mathbf{b}$, the desired solution is $\bar{\mathbf{x}} = [-3\ \ 2\ \ 0\ \ 1]^T$. ■

Potential Pitfalls

-Division by zero: May occur in the forward elimination steps.

-Round-off error: Prone to round-off errors.

Improvements

Increase the number of significant digits

- Decreases round off error
- Does not avoid division by zero

Gaussian Elimination with Pivoting

- Avoids division by zero
- Reduces round off error

LU Decomposition
(Triangular Factorization)

LU Decomposition

A non-singular matrix $[A]$ has a traingular factorization if it can be expressed as

$$[A] = [L][U]$$

where

$[L]$ = lower triangular martix

$[U]$ = upper triangular martix

LU Decomposition

Method: [A] Decompose to [L] and [U]

$$[A] = [L][U] = \begin{bmatrix} 1 & 0 & 0 \\ \ell_{21} & 1 & 0 \\ \ell_{31} & \ell_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

[U] is the same as the coefficient matrix at the end of the forward elimination step.

[L] is obtained using the *multipliers* that were used in the forward elimination process

Example

Given $\mathbf{A} = \begin{pmatrix} 4 & 2 & 3 \\ -3 & 1 & 4 \\ 2 & 4 & 5 \end{pmatrix}$. Find matrices **L** and **U** so that **LU = A**.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 4 & 2 & 3 \\ -3 & 1 & 4 \\ 2 & 4 & 5 \end{pmatrix} = \begin{pmatrix} 4 & 2 & 3 \\ -3 & 1 & 4 \\ 2 & 4 & 5 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ -\frac{3}{4} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 4 & 2 & 3 \\ 0 & \frac{5}{2} & \frac{25}{4} \\ 2 & 4 & 5 \end{pmatrix} = \begin{pmatrix} 4 & 2 & 3 \\ -3 & 1 & 4 \\ 2 & 4 & 5 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ -\frac{3}{4} & 1 & 0 \\ \frac{1}{2} & 0 & 1 \end{pmatrix} \begin{pmatrix} 4 & 2 & 3 \\ 0 & \frac{5}{2} & \frac{25}{4} \\ 0 & 3 & \frac{7}{2} \end{pmatrix} = \begin{pmatrix} 4 & 2 & 3 \\ -3 & 1 & 4 \\ 2 & 4 & 5 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ -\frac{3}{4} & 1 & 0 \\ \frac{1}{2} & \frac{6}{5} & 1 \end{pmatrix} \begin{pmatrix} 4 & 2 & 3 \\ 0 & \frac{5}{2} & \frac{25}{4} \\ 0 & 0 & -4 \end{pmatrix} = \begin{pmatrix} 4 & 2 & 3 \\ -3 & 1 & 4 \\ 2 & 4 & 5 \end{pmatrix}$$

LU Decomposition

Given $[A][X]=[C]$

Decompose $[\mathrm{A}]$ into $[L]$ and $[U]$ $\Rightarrow [L][U][X]=[C]$

Define $[Z]=[U][X]$

Then solve $[\mathrm{L}][\mathrm{Z}]=[C]$ for $[Z]$

And then solve $[U][\mathrm{X}]=[Z]$ for $[X]$

LU Decomposition

Example: Solving simultaneous linear equations using LU Decomposition

Solve the following set of linear equations using LU Decomposition

$$\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}$$

Using the procedure for finding the [*L*] and [*U*] matrices

$$[A]=[L][U]=\begin{bmatrix} 1 & 0 & 0 \\ 2.56 & 1 & 0 \\ 5.76 & 3.5 & 1 \end{bmatrix} \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix}$$

LU Decomposition

Example: Solving simultaneous linear equations using LU Decomposition

Set $[L][Z]=[C]$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2.56 & 1 & 0 \\ 5.76 & 3.5 & 1 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ 177.2 \\ 279.2 \end{bmatrix}$$

Solve for $[Z]$

$$[Z]=\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ -96.21 \\ 0.735 \end{bmatrix}$$

LU Decomposition

Example: Solving simultaneous linear equations using LU Decomposition

Set $[U][X]=[Z]$

$$\begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 106.8 \\ -96.21 \\ 0.735 \end{bmatrix}$$

Solve for $[X]$

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0.2900 \\ 19.70 \\ 1.050 \end{bmatrix}$$

Factorization with Pivoting

Given $A = \begin{pmatrix} 1 & 2 & 6 \\ 4 & 8 & -1 \\ -2 & 3 & 5 \end{pmatrix}$. Can **A** be factored **A = LU**?

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 6 \\ 4 & 8 & -1 \\ -2 & 3 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 6 \\ 4 & 8 & -1 \\ -2 & 3 & 5 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 6 \\ 0 & 0 & -25 \\ -2 & 3 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 6 \\ 4 & 8 & -1 \\ -2 & 3 & 5 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 6 \\ 0 & 0 & -25 \\ 0 & 7 & 17 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 6 \\ 4 & 8 & -1 \\ -2 & 3 & 5 \end{pmatrix}$$

Factorization with Pivoting

- **Theorem.** If **A** is a nonsingular matrix, then there exists a permutation matrix **P** so that **PA** has an LU-factorization

  PA = **LU**.
- **Theorem (PA = LU; Factorization with Pivoting).** Given that **A** is nonsingular. The solution **X** to the linear system **AX=B** , is found in four steps:

  1. Construct the matrices **L**,**U** and **P** .
  2. Compute the column vector **PB** .
  3. Solve **LY=PB** for **Y** using forward substitution.
  4. Solve **UX=Y** for **X** using back substitution.

Example 3.22. Show that the following matrix cannot be factored directly as $A = LU$:

$$A = \begin{bmatrix} 1 & 2 & 6 \\ 4 & 8 & -1 \\ -2 & 3 & 5 \end{bmatrix}.$$

$$PA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 6 \\ 4 & 8 & -1 \\ -2 & 3 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 6 \\ -2 & 3 & 5 \\ 4 & 8 & -1 \end{bmatrix}$$

$$\begin{array}{l} \text{pivot} \rightarrow \\ m_{21} = -2 \\ m_{31} = 4 \end{array} \begin{bmatrix} \underline{1} & 2 & 6 \\ -2 & 3 & 5 \\ 4 & 8 & -1 \end{bmatrix}$$

$$\begin{array}{l} \\ \text{pivot} \rightarrow \\ m_{32} = 0 \end{array} \begin{bmatrix} 1 & 2 & 6 \\ 0 & \underline{7} & 17 \\ 0 & 0 & -25 \end{bmatrix} = U$$

$$[L] = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix}$$

Is *LU Decomposition* better or faster than *Gauss Elimination*?

Let's look at computational time.

n = number of equations

To decompose [A], time is proportional to $\frac{n^3}{3}$

To solve $[U][X] = [C]$ and $[\mathrm{L}][\mathrm{Z}] = [\mathrm{C}]$
time proportional to $\frac{n^2}{2}$

Total computational time for LU Decomposition is proportional to

$$\frac{n^3}{3} + 2(\frac{n^2}{2}) \quad \text{or} \quad \frac{n^3}{3} + n^2$$

Gauss Elimination computation time is proportional to

$$\frac{n^3}{3} + \frac{n^2}{2}$$

How is this better?

LU Decomposition

What about a situation where the [C] vector changes?

In LU Decomposition, LU decomposition of [A] is independent of the [C] vector, therefore it only needs to be done once.

Let m = the number of times the [C] vector changes

The computational times are proportional to

Gauss Elimination = $\mathrm{m}(\frac{\mathrm{n}^3}{3} + \frac{\mathrm{n}^2}{2})$ LU decomposition = $\frac{\mathrm{n}^3}{3} + \mathrm{m}(\mathrm{n}^2)$

Consider a 100 equation set with 50 right hand side vectors

LU Decomposition = 8.33×10^5 Gauss Elimination = 1.69×10^7

Simultaneous Linear Equations: Iterative Methods

Jacobi and Gauss-Seidel Method

Example 3.26. Consider the system of equations

$$(1)\qquad \begin{aligned} 4x - y + z &= 7 \\ 4x - 8y + z &= -21 \\ -2x + y + 5z &= 15. \end{aligned}$$

These equations can be written in the form

$$(2)\qquad \begin{aligned} x &= \frac{7 + y - z}{4} \\ y &= \frac{21 + 4x + z}{8} \\ z &= \frac{15 + 2x - y}{5}. \end{aligned}$$

$$x_{k+1} = \frac{7 + y_k - z_k}{4}$$
$$y_{k+1} = \frac{21 + 4x_k + z_k}{8}$$
$$z_{k+1} = \frac{15 + 2x_k - y_k}{5}.$$

-Algebraically solve each linear equation for x_i

-Assume an initial guess $(x_0, y_0, z_0) = (1, 2, 2)$

-Solve for each x_i and repeat

-Check if error is within a pre-specified tolerance.

Table 3.2 Convergent Jacobi Iteration for the Linear System (1)

k	x_k	y_k	z_k
0	1.0	2.0	2.0
1	1.75	3.375	3.0
2	1.84375	3.875	3.025
3	1.9625	3.925	2.9625
4	1.99062500	3.97656250	3.00000000
5	1.99414063	3.99531250	3.00093750
⋮	⋮	⋮	⋮
15	1.99999993	3.99999985	2.99999993
⋮	⋮	⋮	⋮
19	2.00000000	4.00000000	3.00000000

Jacobi

$$x_{k+1} = \frac{7 + y_k - z_k}{4}$$
$$y_{k+1} = \frac{21 + 4x_k + z_k}{8}$$
$$z_{k+1} = \frac{15 + 2x_k - y_k}{5}.$$

Gauss-Seidel

$$x_{k+1} = \frac{7 + y_k - z_k}{4}$$
$$y_{k+1} = \frac{21 + 4x_{k+1} + z_k}{8}$$
$$z_{k+1} = \frac{15 + 2x_{k+1} - y_{k+1}}{5}.$$

$(x_0, y_0, z_0) = (1, 2, 2)$

Table 3.2 Convergent Jacobi Iteration for the Linear System (1)

k	x_k	y_k	z_k
0	1.0	2.0	2.0
1	1.75	3.375	3.0
2	1.84375	3.875	3.025
3	1.9625	3.925	2.9625
4	1.99062500	3.97656250	3.00000000
5	1.99414063	3.99531250	3.00093750
⋮	⋮	⋮	⋮
15	1.99999993	3.99999985	2.99999993
⋮	⋮	⋮	⋮
19	2.00000000	4.00000000	3.00000000

Table 3.4 Convergent Gauss-Seidel Iteration for the System (1)

k	x_k	y_k	z_k
0	1.0	2.0	2.0
1	1.75	3.75	2.95
2	1.95	3.96875	2.98625
3	1.995625	3.99609375	2.99903125
⋮	⋮	⋮	⋮
8	1.99999983	3.99999988	2.99999996
9	1.99999998	3.99999999	3.00000000
10	2.00000000	4.00000000	3.00000000

Algorithm

A set of *n* equations and *n* unknowns:

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + ... + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + ... + a_{2n}x_n = b_2$$
$$\vdots \qquad \vdots$$
$$a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + ... + a_{nn}x_n = b_n$$

If: the diagonal elements are non-zero

Rewrite each equation solving for the corresponding unknown

ex:

First equation, solve for x_1

Second equation, solve for x_2

Algorithm

Rewriting each equation

$$x_1 = \frac{c_1 - a_{12}x_2 - a_{13}x_3 \ldots\ldots - a_{1n}x_n}{a_{11}}$$ From Equation 1

$$x_2 = \frac{c_2 - a_{21}x_1 - a_{23}x_3 \ldots\ldots - a_{2n}x_n}{a_{22}}$$ From equation 2

$$\vdots \qquad \vdots \qquad \vdots$$

$$x_{n-1} = \frac{c_{n-1} - a_{n-1,1}x_1 - a_{n-1,2}x_2 \ldots\ldots - a_{n-1,n-2}x_{n-2} - a_{n-1,n}x_n}{a_{n-1,n-1}}$$ From equation n-1

$$x_n = \frac{c_n - a_{n1}x_1 - a_{n2}x_2 - \ldots\ldots - a_{n,n-1}x_{n-1}}{a_{nn}}$$ From equation n

$$\begin{array}{llllll}
a_{11}x_1 + a_{12}x_2 & + \cdots + a_{1j}x_j & + \cdots + & a_{1N}x_N & = b_1 \\
a_{21}x_1 + a_{22}x_2 & + \cdots + a_{2j}x_j & + \cdots + & a_{2N}x_N & = b_2 \\
\vdots \quad \vdots & \vdots & & \vdots & \vdots \\
a_{j1}x_1 + a_{j2}x_2 & + \cdots + a_{jj}x_j & + \cdots + & a_{jN}x_N & = b_j \\
\vdots \quad \vdots & \vdots & & \vdots & \vdots \\
a_{N1}x_1 + a_{N2}x_2 & + \cdots + a_{Nj}x_j & + \cdots + & a_{NN}x_N & = b_N.
\end{array}$$

Jacobi iteration:

$$x_j^{(k+1)} = \frac{b_j - a_{j1}x_1^{(k)} - \cdots - a_{jj-1}x_{j-1}^{(k)} - a_{jj+1}x_{j+1}^{(k)} - \cdots - a_{jN}x_N^{(k)}}{a_{jj}} \tag{10}$$

for $j = 1, 2, \ldots, N$.

Gauss-Seidel iteration:

$$x_j^{(k+1)} = \frac{b_j - a_{j1}x_1^{(k+1)} - \cdots - a_{jj-1}x_{j-1}^{(k+1)} - a_{jj+1}x_{j+1}^{(k)} - \cdots - a_{jN}x_N^{(k)}}{a_{jj}} \tag{11}$$

for $j = 1, 2, \ldots, N$.

Stopping criterion

Absolute Relative Error

$$\left|\varepsilon_a\right|_i = \left|\frac{x_i^{new} - x_i^{old}}{x_i^{new}}\right| \times 100$$

The iterations are stopped when the absolute relative error is less than a prespecified tolerance for all unknowns.

Given the system of equations

$$12x_1 + 3x_2 - 5x_3 = 1$$
$$x_1 + 5x_2 + 3x_3 = 28$$
$$3x_1 + 7x_2 + 13x_3 = 76$$

With an initial guess of

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Rewriting each equation

$$\begin{bmatrix} 12 & 3 & -5 \\ 1 & 5 & 3 \\ 3 & 7 & 13 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 28 \\ 76 \end{bmatrix}$$

$$x_1 = \frac{1 - 3x_2 + 5x_3}{12} \qquad x_1 = \frac{1 - 3(0) + 5(1)}{12} = 0.50000$$

$$x_2 = \frac{28 - x_1 - 3x_3}{5} \qquad x_2 = \frac{28 - (0.5) - 3(1)}{5} = 4.9000$$

$$x_3 = \frac{76 - 3x_1 - 7x_2}{13} \qquad x_3 = \frac{76 - 3(0.50000) - 7(4.9000)}{13} = 3.0923$$

Initial guess

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

After Iteration #1

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0.5000 \\ 4.9000 \\ 3.0923 \end{bmatrix}$$

The absolute relative error

$$\left|\in_a\right|_1 = \left|\frac{0.50000 - 1.0000}{0.50000}\right| \times 100 = 67.662\%$$

$$\left|\in_a\right|_2 = \left|\frac{4.9000 - 0}{4.9000}\right| \times 100 = 100.00\%$$

$$\left|\in_a\right|_3 = \left|\frac{3.0923 - 1.0000}{3.0923}\right| \times 100 = 67.662\%$$

The maximum absolute relative error after the first iteration is 100%

Repeating more iterations, the following values are obtained

Iteration	a_1	$\lvert\varepsilon_a\rvert_1$	a_2	$\lvert\varepsilon_a\rvert_2$	a_3	$\lvert\varepsilon_a\rvert_3$
1	0.50000	67.662	4.900	100.00	3.0923	67.662
2	0.14679	240.62	3.7153	31.887	3.8118	18.876
3	0.74275	80.23	3.1644	17.409	3.9708	4.0042
4	0.94675	21.547	3.0281	4.5012	3.9971	0.65798
5	0.99177	4.5394	3.0034	0.82240	4.0001	0.07499
6	0.99919	0.74260	3.0001	0.11000	4.0001	0.00000

The solution obtained $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0.99919 \\ 3.0001 \\ 4.0001 \end{bmatrix}$

the exact solution $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$

$$\begin{aligned} 4x - y + z &= 7 \\ 4x - 8y + z &= -21 \\ -2x + y + 5z &= 15. \end{aligned}$$

Example 3.27. Let the linear system (1) be rearranged as follows:

$$\begin{aligned} -2x + y + 5z &= 15 \\ 4x - 8y + z &= -21 \\ 4x - y + z &= 7. \end{aligned} \tag{4}$$

Table 3.3 Divergent Jacobi Iteration for the Linear System (4)

k	x_k	y_k	z_k
0	1.0	2.0	2.0
1	−1.5	3.375	5.0
2	6.6875	2.5	16.375
3	34.6875	8.015625	−17.25
4	−46.617188	17.8125	−123.73438
5	−307.929688	−36.150391	211.28125
6	502.62793	−124.929688	1202.56836
⋮	⋮	⋮	⋮

What went wrong?

Even though done correctly, the answer is not converging to the correct answer

This example illustrates a pitfall of Jacobi/ Gauss-Siedel method: not all systems of equations will converge.

Is there a fix?

Theorem 3.15 (Jacobi Iteration). Suppose that $\boldsymbol{A}$ is a strictly diagonally dominant matrix. Then $\boldsymbol{AX} = \boldsymbol{B}$ has a unique solution $\boldsymbol{X} = \boldsymbol{P}$. Iteration using formula (10) will produce a sequence of vectors $\{\boldsymbol{P}_k\}$ that will converge to $\boldsymbol{P}$ for any choice of the starting vector $\boldsymbol{P}_0$.

Diagonally dominant: [A] in [A] [X] = [C] is diagonally dominant if:

$$|\mathbf{a}_{ii}| > \sum_{\substack{\mathbf{j}=1 \\ \mathbf{j}\neq i}}^{\mathbf{n}} |\mathbf{a}_{\mathbf{ij}}|$$

The coefficient on the diagonal must be greater than the sum of the other coefficients in that row.

$$\begin{aligned} 4x - y + z &= 7 \\ 4x - 8y + z &= -21 \\ -2x + y + 5z &= 15. \end{aligned}$$

In row 1: $|-2| < |1| + |5|$

In row 2: $|-8| > |4| + |1|$

In row 3: $|1| < |4| + |-1|$.

$$\begin{aligned} -2x + y + 5z &= 15 \\ 4x - 8y + z &= -21 \\ 4x - y + z &= 7. \end{aligned}$$

In row 1: $|4| > |-1| + |1|$

In row 2: $|-8| > |4| + |1|$

In row 3: $|5| > |-2| + |1|$.

Table 3.2 Convergent Jacobi Iteration for the Linear System (1)

k	x_k	y_k	z_k
0	1.0	2.0	2.0
1	1.75	3.375	3.0
2	1.84375	3.875	3.025
3	1.9625	3.925	2.9625
4	1.99062500	3.97656250	3.00000000
5	1.99414063	3.99531250	3.00093750
⋮	⋮	⋮	⋮
15	1.99999993	3.99999985	2.99999993
⋮	⋮	⋮	⋮
19	2.00000000	4.00000000	3.00000000

Table 3.3 Divergent Jacobi Iteration for the Linear System (4)

k	x_k	y_k	z_k
0	1.0	2.0	2.0
1	−1.5	3.375	5.0
2	6.6875	2.5	16.375
3	34.6875	8.015625	−17.25
4	−46.617188	17.8125	−123.73438
5	−307.929688	−36.150391	211.28125
6	502.62793	−124.929688	1202.56836
⋮	⋮	⋮	⋮