# Solution of Linear Systems

A linear combination of the variables  $x_1, x_2, \ldots, x_N$  is a sum

$$(1) a_1x_1 + a_2x_2 + \dots + a_Nx_N$$

where  $a_k$  is the coefficient of  $x_k$  for k = 1, 2, ..., N.

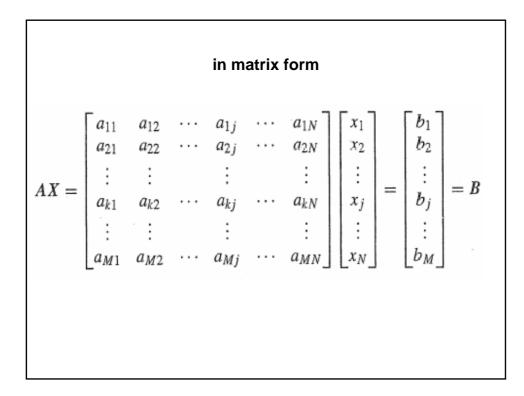
A linear equation in  $x_1, x_2, \ldots, x_N$  is obtained by requiring the linear combination in (1) to take on a prescribed value *b*; that is,

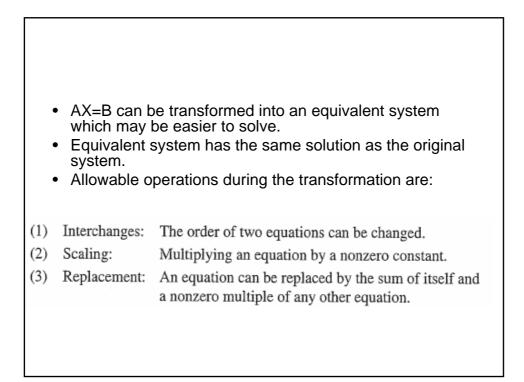
(2)  $a_1x_1 + a_2x_2 + \dots + a_Nx_N = b.$ 

(3)

Systems of linear equations arise frequently, and if M equations in N unknowns are given, we write

 $a_{11}x_{1} + a_{12}x_{2} + \dots + a_{1N}x_{N} = b_{1}$   $a_{21}x_{1} + a_{22}x_{2} + \dots + a_{2N}x_{N} = b_{2}$   $\vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots$   $a_{k1}x_{1} + a_{k2}x_{2} + \dots + a_{kN}x_{N} = b_{k}$   $\vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots$   $a_{M1}x_{1} + a_{M2}x_{2} + \dots + a_{MN}x_{N} = b_{M}.$ 

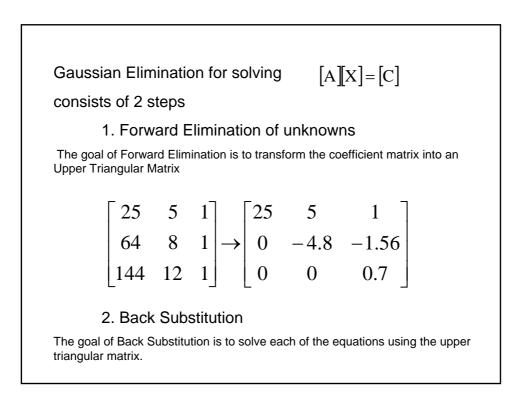




Example 3.15. Find the parabola  $y = A + Bx + Cx^2$  that passes through the three points (1, 1), (2, −1), and (3, 1). For each point we obtain an equation relating the value of x to the value of y. The result is the linear system A + B + C = 1 at (1, 1) A + 2B + 4C = -1 at (2, -1)(4)A + 3B + 9C = 1 at (3, 1). The variable A is eliminated from the second and third equations by subtracting the first equation from them. This is an application of the replacement transformation (3), and the resulting equivalent linear system is A + B + C = 1(5) B + 3C = -22B + 8C = 0.The variable B is eliminated from the third equation in (5) by subtracting from it two times the second equation. We arrive at the equivalent upper-triangular system: A + B + C = 1(6) B + 3C = -22C = 4. The back-substitution algorithm is now used to find the coefficients C = 4/2 = 2, B =-2 - 3(2) = -8, and A = 1 - (-8) - 2 = 7, and the equation of the parabola is

.

 $y = 7 - 8x + 2x^2$ .



### **Gaussian Elimination**

The augmented matrix is [A|B] and the system AX = B is represented as follows:

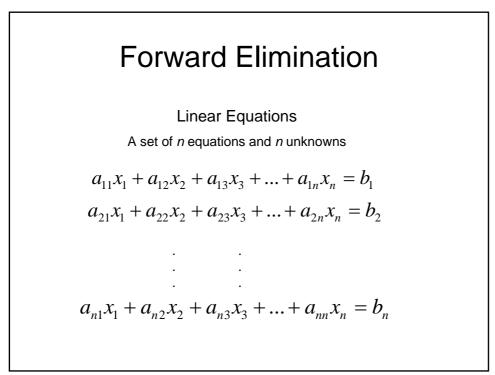
(7)  $[A|B] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1N} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2N} & b_2 \\ \vdots & \vdots & & \vdots & \\ a_{N1} & a_{N2} & \cdots & a_{NN} & b_N \end{bmatrix}.$ 

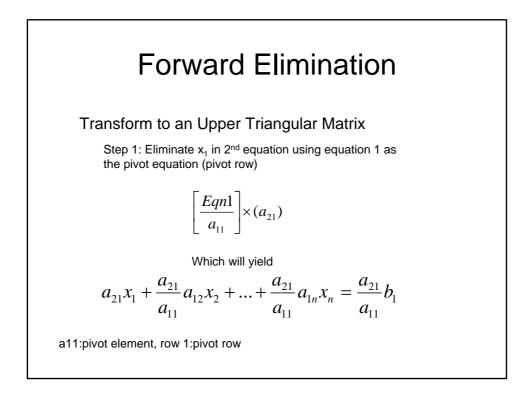
The system AX = B, with augmented matrix given in (7), can be solved by performing row operations on the augmented matrix [A|B]. The variables  $x_k$  are placeholders for the coefficients and can be omitted until the end of the calculation.

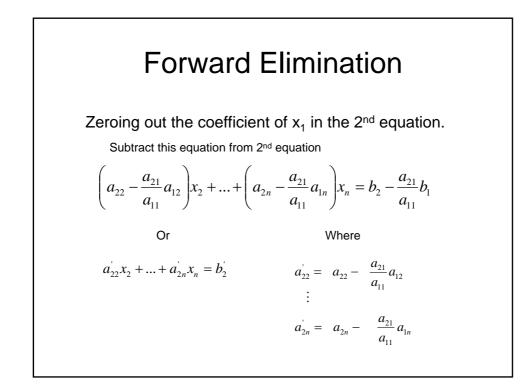
**Theorem 3.8 (Elementary Row Operations).** The following operations applied to the augmented matrix (7) yield an equivalent linear system.

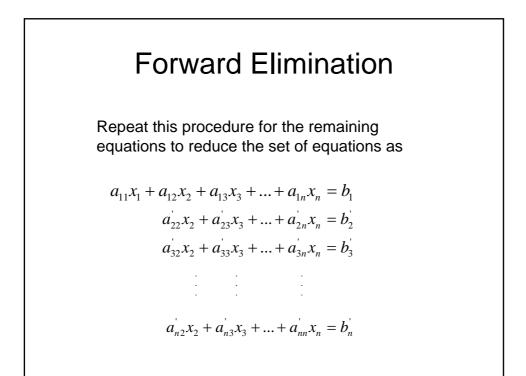
- (8) Interchanges: The order of two rows can be changed.
- (9) Scaling: Multiplying a row by a nonzero constant.
- (10) Replacement: The row can be replaced by the sum of that row and a nonzero multiple of any other row; that is:  $row_r = row_r - m_{rp} \times row_p.$

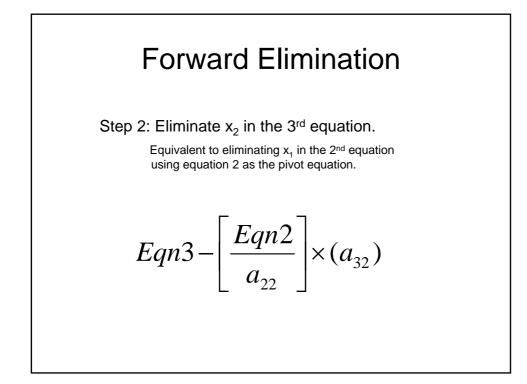
Example 3.16	$x_1 + 2x_2 + x_3 + 4x_4 = 13$ $2x_1 + 0x_2 + 4x_3 + 3x_4 = 28$ $4x_1 + 2x_2 + 2x_3 + x_4 = 20$ $-3x_1 + x_2 + 3x_3 + 2x_4 = 6.$
$ \begin{array}{c c} \text{pivot} \rightarrow \\ m_{21} = 2 \\ m_{31} = 4 \\ m_{41} = -3 \end{array} \begin{bmatrix} 1 & 2 & 1 & 4 & 13 \\ 2 & 0 & 4 & 3 & 28 \\ 4 & 2 & 2 & 1 & 20 \\ -3 & 1 & 3 & 2 & 6 \end{bmatrix} $	
$ \begin{array}{c ccccc} pivot \rightarrow \\ m_{32} = 1.5 \\ m_{42} = -1.75 \end{array} \begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & -4 & 2 & -5 \\ 0 & -6 & -2 & -15 \\ 0 & 7 & 6 & 14 \end{bmatrix} - $	$\begin{bmatrix} 13 \\ 2 \\ 32 \\ 45 \end{bmatrix}$
$ \begin{array}{c} \text{pivot} \rightarrow \\ m_{43} = -1.9 \end{array} \begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & -4 & 2 & -5 \\ 0 & 0 & -5 & -7.5 \\ 0 & 0 & 9.5 & 5.25 \end{bmatrix} - 43 $	13 2 35 3.5
$\begin{bmatrix} 1 & 2 & 1 & 4 &   & 13 \\ 0 & -4 & 2 & -5 & 2 \\ 0 & 0 & -5 & -7.5 & -35 \\ 0 & 0 & 0 & -9 &   & -18 \end{bmatrix} \qquad x_4$	$= 2,  x_3 = 4,  x_2 = -1,  x_1 = 3.$

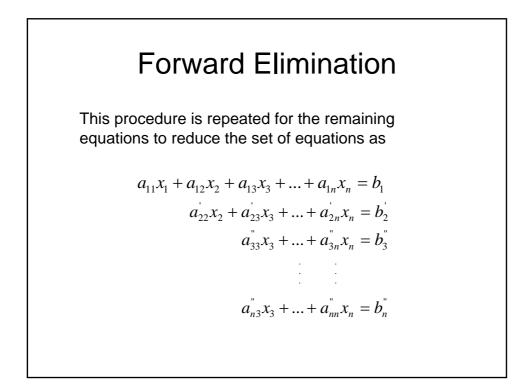


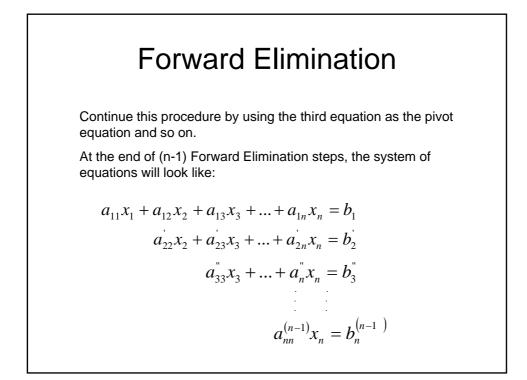


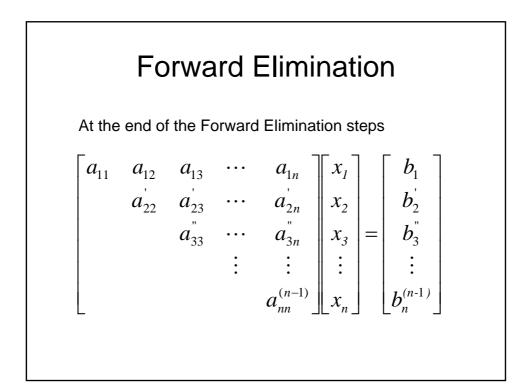


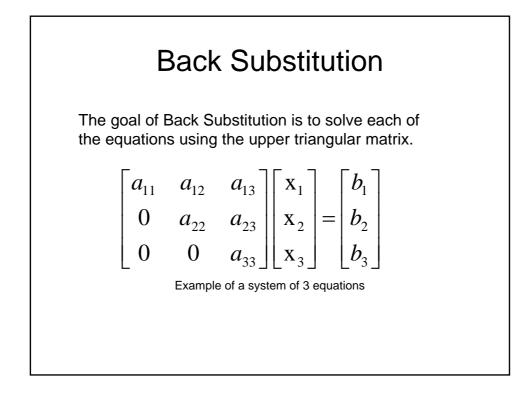


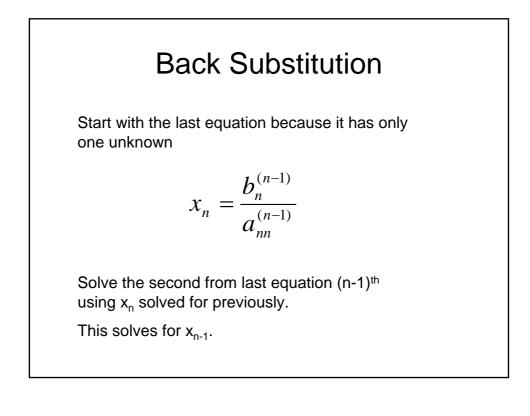


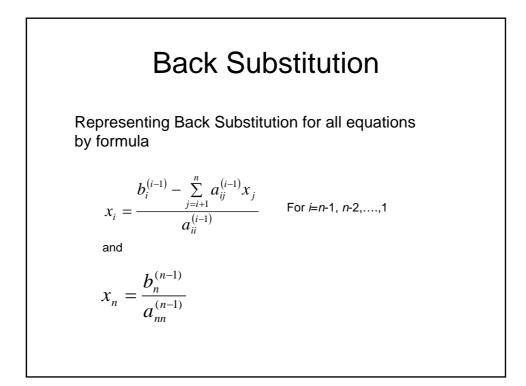


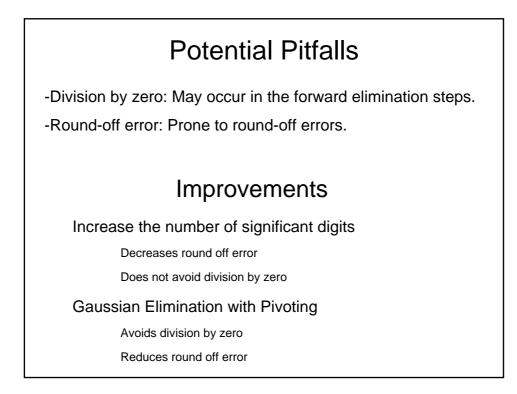


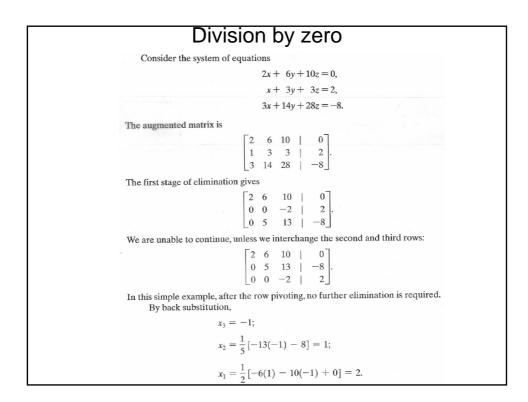








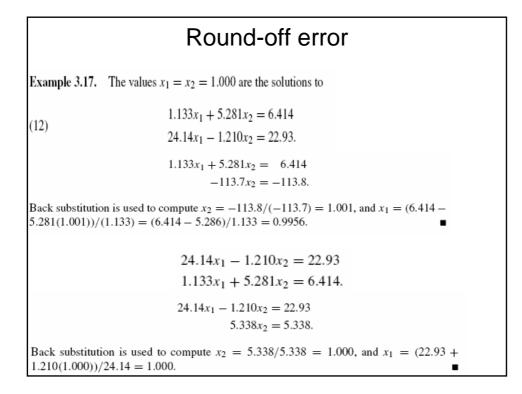


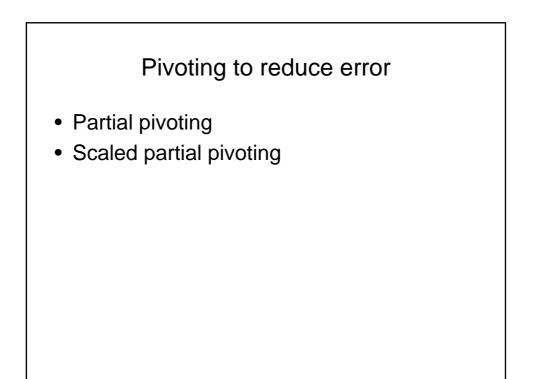


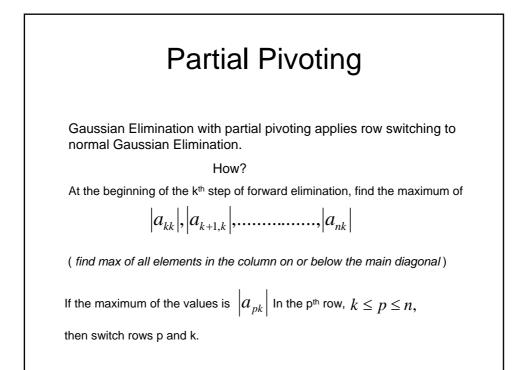
## **Trivial pivoting**

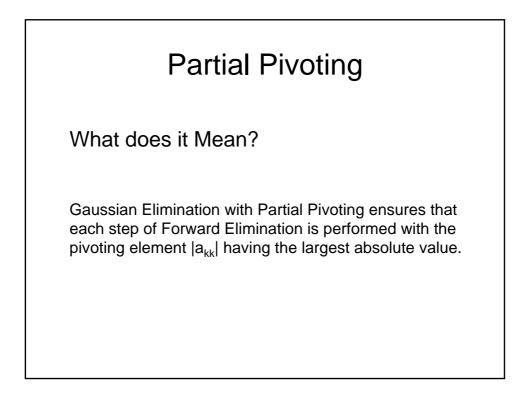
#### Pivoting to Avoid $a_{pp}^{(p)} = 0$

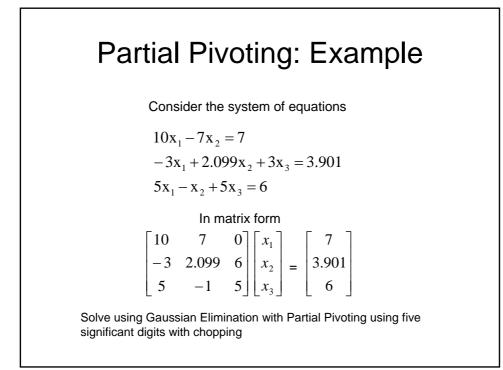
If  $a_{pp}^{(p)} = 0$ , row p cannot be used to eliminate the elements in column p below the main diagonal. It is necessary to find row k, where  $a_{kp}^{(p)} \neq 0$  and k > p, and then interchange row p and row k so that a nonzero pivot element is obtained. This process is called *pivoting*, and the criterion for deciding which row to choose is called a *pivoting* strategy. The *trivial pivoting* strategy is as follows. If  $a_{pp}^{(p)} \neq 0$ , do not switch rows. If  $a_{pp}^{(p)} = 0$ , locate the first row below p in which  $a_{kp}^{(p)} \neq 0$  and switch rows k and p. This will result in a new element  $a_{pp}^{(p)} \neq 0$ , which is a nonzero pivot element.

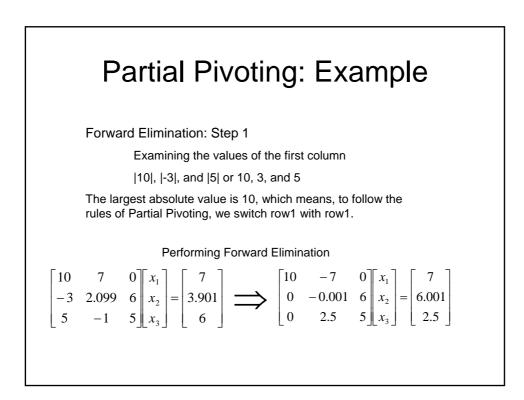


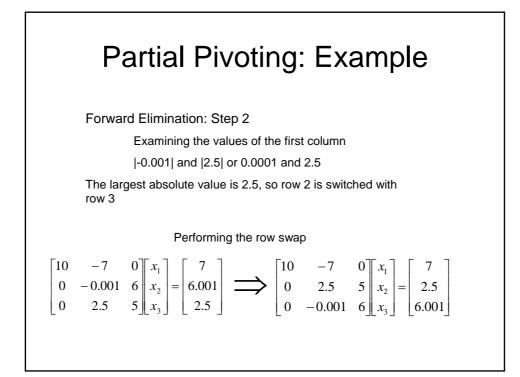


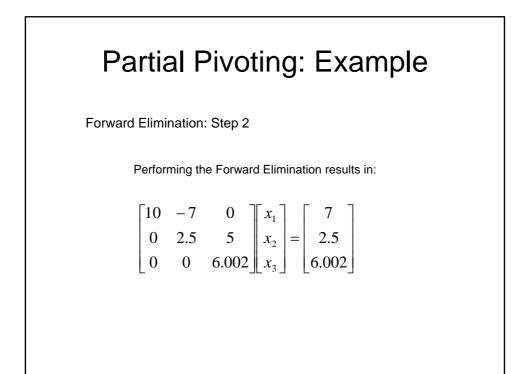


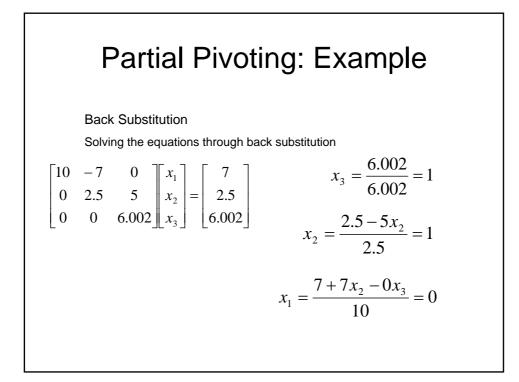












#### Scaled partial pivoting

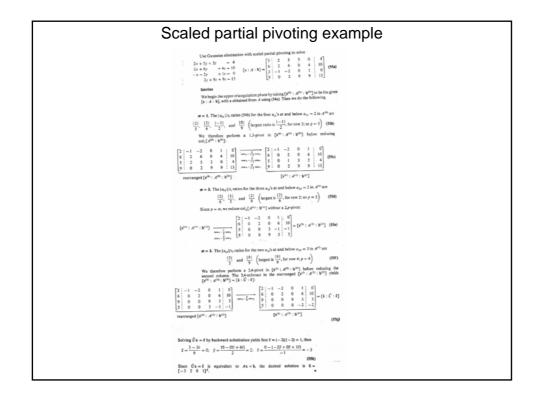
tations could result in an erroneous answer. The technique of *scaled partial pivoting* or equilibrating can be used to further reduce the effect of error propagation. In scaled partial pivoting we search all the elements in column p that lie on or below the main diagonal for the one that is largest relative to the entries in its row. First search rows p through N for the largest element in magnitude in each row, say  $s_r$ :

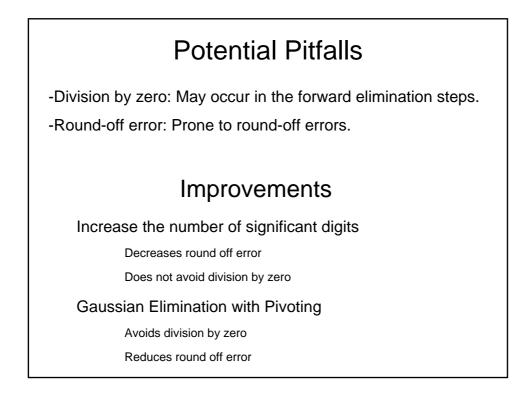
(13)  $s_r = \max\{|a_{rp}|, |a_{rp+1}|, \dots, |a_{rN}|\}$  for  $r = p, p+1, \dots, N$ .

The pivotal row k is determined by finding

(14) 
$$\frac{|a_{kp}|}{s_k} = \max\left\{\frac{|a_{pp}|}{s_p}, \frac{|a_{p+1p}|}{s_{p+1}}, \dots, \frac{|a_{Np}|}{s_N}\right\}.$$

Now interchange row p and k, unless p = k. Again, this pivoting process is designed to keep the relative magnitudes of the elements in the matrix U in Theorem 3.9 the same as those in the original coefficient matrix A.





# LU Decomposition (Triangular Factorization)

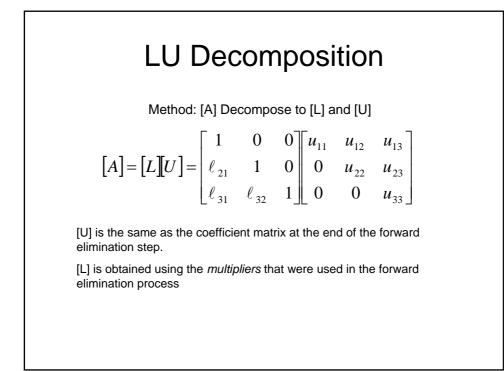
# LU Decomposition

A non-singular matrix  $\left[ A\right]$  has a traingular factorization if it can be expressed as

[A] = [L][U]

where

- [L] = lower triangular martix
- [U] = upper triangular martix

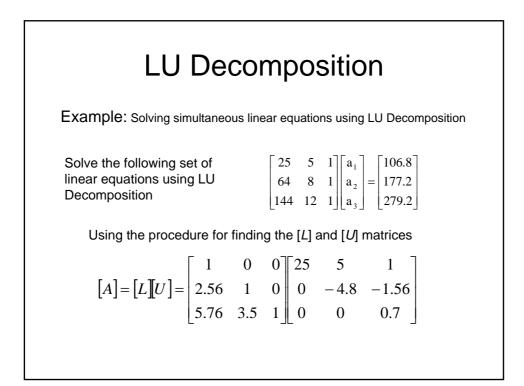


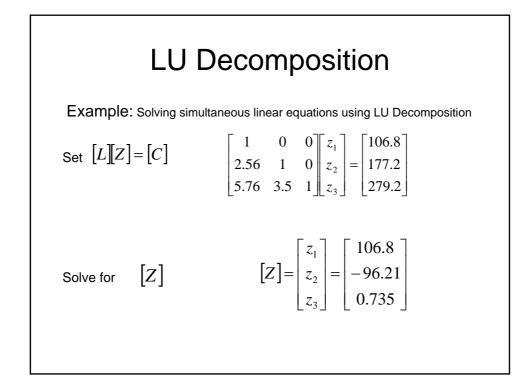
Given $\mathbf{A} = \begin{pmatrix} 4 & 2 & 3 \\ -3 & 1 & 4 \\ 2 & 4 & 5 \end{pmatrix}$ . Find matrices L and U so that LU = A.	
$ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 4 & 2 & 3 \\ -3 & 1 & 4 \\ 2 & 4 & 5 \end{pmatrix} = \begin{pmatrix} 4 & 2 & 3 \\ -3 & 1 & 4 \\ 2 & 4 & 5 \end{pmatrix} $ $ \begin{pmatrix} 1 & 0 & 0 \\ -\frac{2}{4} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 4 & 2 & 3 \\ 0 & \frac{5}{2} & \frac{25}{4} \\ 2 & 4 & 5 \end{pmatrix} = \begin{pmatrix} 4 & 2 & 3 \\ -3 & 1 & 4 \\ 2 & 4 & 5 \end{pmatrix} $ $ \begin{pmatrix} 1 & 0 & 0 \\ -\frac{2}{4} & 1 & 0 \\ \frac{1}{2} & 0 & 1 \end{pmatrix} \begin{pmatrix} 4 & 2 & 3 \\ 0 & \frac{5}{2} & \frac{25}{4} \\ 0 & 3 & \frac{7}{2} \end{pmatrix} = \begin{pmatrix} 4 & 2 & 3 \\ -3 & 1 & 4 \\ 2 & 4 & 5 \end{pmatrix} $ $ \begin{pmatrix} 1 & 0 & 0 \\ -\frac{2}{4} & 1 & 0 \\ \frac{1}{2} & 0 & 1 \end{pmatrix} \begin{pmatrix} 4 & 2 & 3 \\ 0 & \frac{5}{2} & \frac{25}{4} \\ 0 & 3 & \frac{7}{2} \end{pmatrix} = \begin{pmatrix} 4 & 2 & 3 \\ -3 & 1 & 4 \\ 2 & 4 & 5 \end{pmatrix} $	

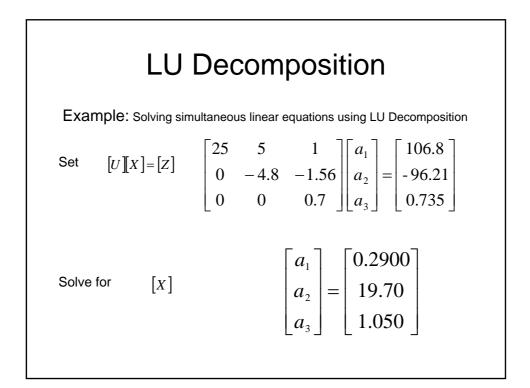
### LU Decomposition

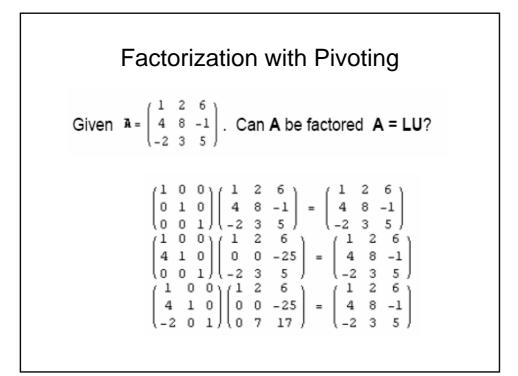
Given [A][X] = [C]Decompose [A] into [L] and  $[U] \Rightarrow [L][U][X] = [C]$ **Define** [Z] = [U][X]Then solve [L][Z] = [C] for [Z]

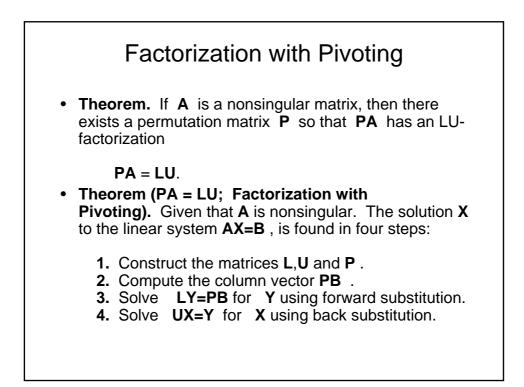
And then solve [U][X] = [Z] for [X]

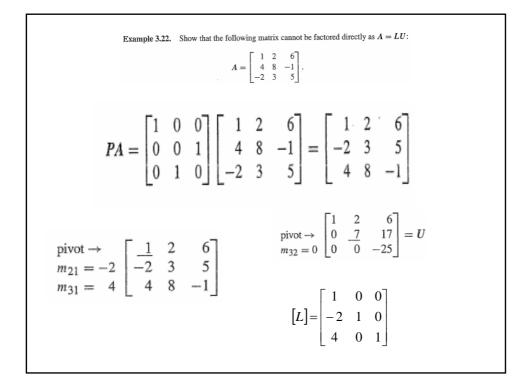


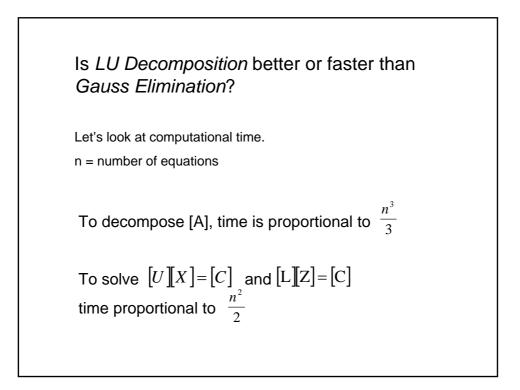


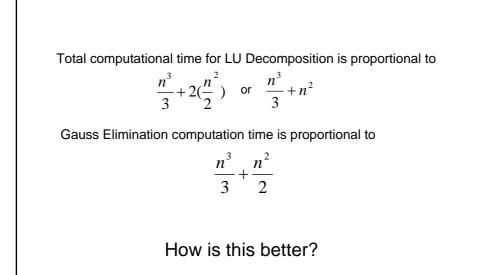


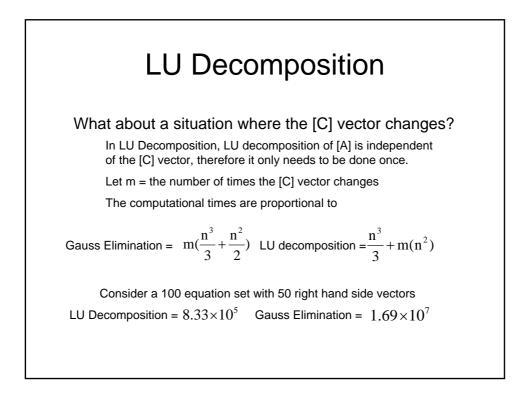






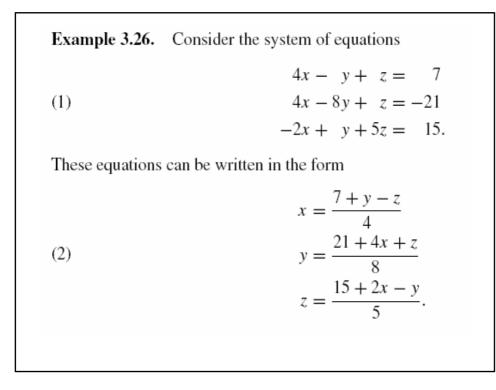






#### Simultaneous Linear Equations: Iterative Methods

Jacobi and Gauss-Seidel Method

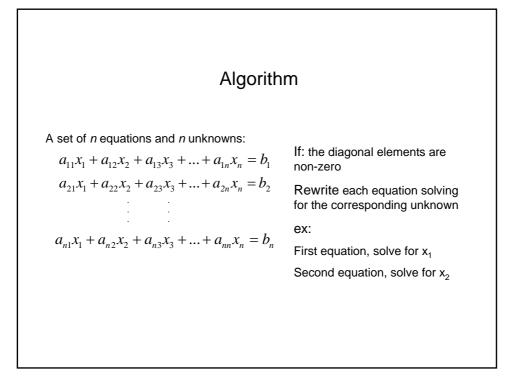


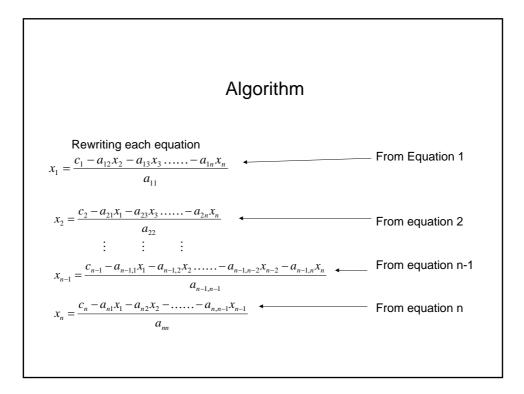
$x_{k+1} = \frac{7 + y_k - z_k}{4}$	-Algebraically solve each linear equation for $\mathbf{x}_{i}$
$21 + 4x_k + z_k$	-Assume an initial guess $(x_0, y_0, z_0) = (1, 2, 2)$
$y_{k+1} =$	-Solve for each x <sub>i</sub> and repeat
$z_{k+1} = \frac{15 + 2x_k - y_k}{5}.$	-Check if error is within a pre-specified tolerance.

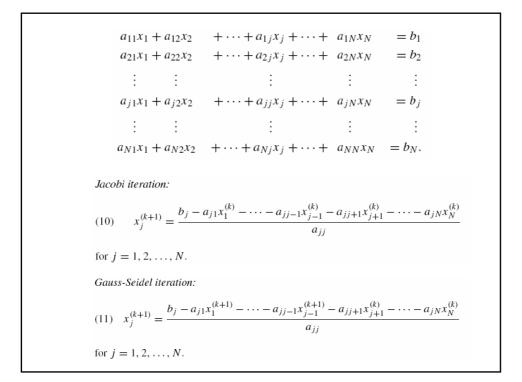
Table 3.2Convergent Jacobi Iteration for the LinearSystem (1)

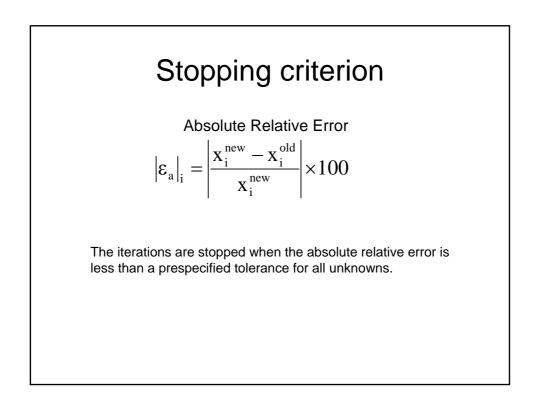
k	x <sub>k</sub>	$y_k$	$z_k$
0	1.0	2.0	2.0
1	1.75	3.375	3.0
2	1.84375	3.875	3.025
3	1.9625	3.925	2.9625
4	1.99062500	3.97656250	3.0000000
5	1.99414063	3.99531250	3.00093750
:	:	:	:
15	1.99999993	3.99999985	2.99999993
:	:	:	:
19	2.00000000	4.00000000	3.0000000

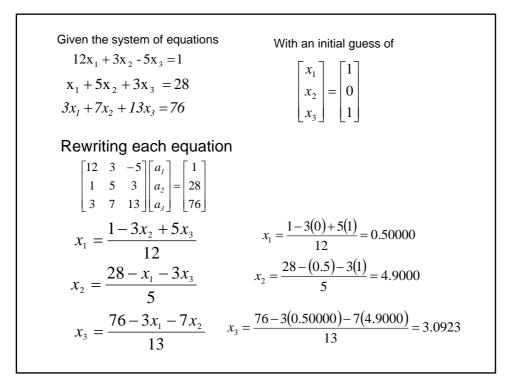
4         1.99062500         3.97656250         3.00000000         .		Jacobi				Gauss-	Seidel	
Table 3.2 Convergent Jacobi Iteration for the Linear System (1)       Table 3.4 Convergent Gauss-Seidel Iteration for the System (1)         k $x_k$ $y_k$ $z_k$ $x_k$ $y_k$ $z_k$ 0       1.0       2.0       2.0       0       1.0       2.0       2.0       0       1.0       2.0       2.0       2.0       1       1.75       3.375       3.0       1       1.75       3.75       2.95       2.95       2.95       2.9625       3       1.955       3.96875       2.98625       3.99609375       2.999031       4       1.99062500       3.97656250       3.00000000 $\vdots$ <	)	$y_{k+1} = \frac{21+4}{3}$	$\frac{x_k + z_k}{8}$	(r- y- r-) = (1	2 2)	$y_{k+1} = \frac{2}{2}$	$\frac{1+4x_{k+1}+}{8}$	
$ \begin{array}{c c c c c c c c c c c c c c c c c c c $				$(x_0, y_0, z_0) = (1$	, 2, 2)			
$ \begin{array}{c c c c c c c c c c c c c c c c c c c $	Syster	m (1)			Syster	n (1)		
1         1.75         3.375         3.0         1         1.75         3.75         2.95           2         1.84375         3.875         3.025         2         1.95         3.96875         2.98625           3         1.9625         3.925         2.9625         3         1.995625         3.99609375         2.999031           4         1.99062500         3.97656250         3.00093750         :	0	1.0	2.0	2.0	0	1.0	2.0	2.0
3         1.9625         3.925         2.9625         3         1.995625         3.99609375         2.990313           4         1.99062500         3.97656250         3.00000000         :<	1	1.75	3.375	3.0	-			
4       1.99062500       3.97656250       3.00000000       .       1.9906250       3.9909975       2.9999995         5       1.99414063       3.99531250       3.00093750       .       <	2	1.84375	3.875	3.025	2	1.95	3.96875	2.98625
5       1.99414063       3.99531250       3.00093750       ::		1.9625	3.925	2.9625	3	1.995625	3.99609375	2.99903125
:         :         :         :         9         1.99999983         3.99999988         2.9999999           15         1.99999993         3.9999985         2.9999993         10         2.00000000         4.00000000         3.000000           :         :         :         :         :         :         :         :         :         :	3	1.99062500	3.97656250	3.00000000				
::         ::         9         1.99999998         3.9999999         3.000000           15         1.99999993         3.99999985         2.99999993         10         2.00000000         4.00000000         3.000000           :: <td< td=""><td>-</td><td></td><td></td><td>2 00002750</td><td>:</td><td>:</td><td>:</td><td>:</td></td<>	-			2 00002750	:	:	:	:
15         1.99999993         3.99999985         2.99999993         10         2.0000000         4.0000000         3.000000           1:         :	4	1.99414063	3.99531250	3.00093750		1 0000000		
	4 5	1.99414063	3.99531250 :	3.00093750	-			
	4 5 :	:	÷	:	9	1.99999998	3.99999999	3.00000000
19 2.00000000 4.0000000 3.0000000	4 5 : 15	:	÷	:	9	1.99999998	3.99999999	
	4 5 : 15	:	÷	:	9	1.99999998	3.99999999	3.00000000

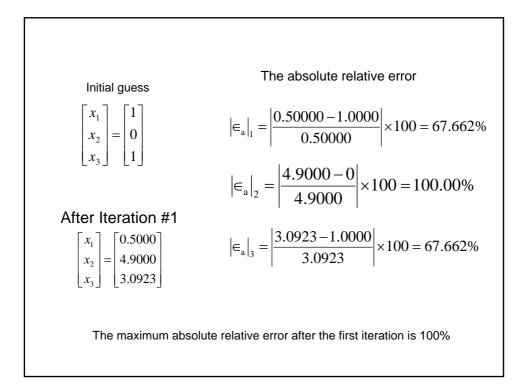












	Iteration	$a_1$	E	<i>a</i> <sub>2</sub>		<i>a</i> <sub>3</sub>	
	licitation	1	$\left \mathcal{E}_{a}\right _{1}$	<sup>42</sup> 2	$\left \mathcal{E}_{a}\right _{2}$	3	$\left  \boldsymbol{\mathcal{E}}_{a} \right _{3}$
	1	0.50000	67.662	4.900	100.00	3.0923	67.662
	2	0.14679	240.62	3.7153	31.887	3.8118	18.876
	3	0.74275	80.23	3.1644	17.409	3.9708	4.0042
	4	0.94675	21.547	3.0281	4.5012	3.9971	0.65798
	5	0.99177	4.5394	3.0034	0.82240	4.0001	0.07499
	6	0.99919	0.74260	3.0001	0.11000	4.0001	0.00000
solu	tion obtai	ned $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$	= 3.000	01			
exac	t solution	[r		L			

4x - y + z = 7 $4x - 8y + z = -21$	1	Example 3.27.	Let the linear system	(1) be rearranged as f	ollows:		
-2x + y + 5z = 15	5.		-2x	+ y + 5z = 15			
		(4) $4x - 8y + z = -21$					
		( )		-y + z = 7.			
			74	y = z = z			
:	Syste	em (4)					
	k	XI	VL	Zŧ			
_		<i>x</i> <sub>k</sub>	<i>y<sub>k</sub></i>	<i>z<sub>k</sub></i>			
	0	1.0	2.0	2.0			
(	0 1						
	0	1.0 -1.5	2.0 3.375	2.0 5.0			
	0 1 2	1.0 -1.5 6.6875	2.0 3.375 2.5	2.0 5.0 16.375			
	0 1 2 3	1.0 -1.5 6.6875 34.6875	2.0 3.375 2.5 8.015625	2.0 5.0 16.375 -17.25			
	0 1 2 3 4	$ \begin{array}{r} 1.0 \\ -1.5 \\ 6.6875 \\ 34.6875 \\ -46.617188 \end{array} $	2.0 3.375 2.5 8.015625 17.8125	2.0 5.0 16.375 -17.25 -123.73438			
	0 1 2 3 4 5	$ \begin{array}{r} 1.0 \\ -1.5 \\ 6.6875 \\ 34.6875 \\ -46.617188 \\ -307.929688 \\ \end{array} $	2.0 3.375 2.5 8.015625 17.8125 -36.150391	$2.0 \\ 5.0 \\ 16.375 \\ -17.25 \\ -123.73438 \\ 211.28125$			

#### What went wrong?

Even though done correctly, the answer is not converging to the correct answer

This example illustrates a pitfall of Jacobi/ Gauss-Siedel method: not all systems of equations will converge.

#### Is there a fix?

**Theorem 3.15** (Jacobi Iteration). Suppose that *A* is a strictly diagonally dominant matrix. Then AX = B has a unique solution X = P. Iteration using formula (10) will produce a sequence of vectors  $\{P_k\}$  that will converge to *P* for any choice of the starting vector  $P_0$ .

Diagonally dominant: [A] in [A] [X] = [C] is diagonally dominant if:

$$\mathbf{a}_{ii} \Big| > \sum_{\substack{\mathbf{j}=1\\\mathbf{j} \neq i}}^{\mathbf{n}} \Big| \mathbf{a}_{ij} \Big|$$

The coefficient on the diagonal must be greater than the sum of the other coefficients in that row.

	4x -	y + z = 7 8y + z = -21 y + 5z = 15				5z = 15 z = -21 z = 7.	
	In row 2:	-2  <    -8  >    1  <  4	4  +  1	In	row 1:  4  : row 2:   - row 3:  5  :	8  >  4  +  1	
Table Syster	3.2 Convergent	Jacobi Iteration for		Tab		Jacobi Iteration for	the Linear
k	x <sub>k</sub>	Уk	zk	k	xk	y <sub>k</sub>	zk
0	1.0	2.0	2.0	0	1.0	2.0	2.0
1	1.75	3.375	3.0	1	-1.5	3.375	5.0
2	1.84375	3.875	3.025	2	6.6875	2.5	16.375
3	1.9625	3.925	2.9625	3	34.6875	8.015625	-17.25
4	1.99062500	3.97656250	3.00000000	4	-46.617188	17.8125	-123.73438
5	1.99414063	3.99531250	3.00093750	5	-307.929688	-36.150391	211.28125
÷	:	:	:	6	502.62793	-124.929688	1202.56836
15	1.99999993	3.99999985	2.99999993	:	:	:	:
÷	:	:	:		•		
19	2.00000000	.4.00000000	3.00000000				
	·		<u>.</u>				